Received: *May 31, 2000,* Accepted: *August 3, 2000*

Hidden superconformal symmetry in M-theory

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Abstract: The bosonic sectors of the eleven-dimensional and IIA supergravity theories are derived as non-linear realisations. The underlying group includes the conformal group, the general linear group and as well as automorphisms of the supersymmetry algebra. We discuss the supersymmetric extension and argue that OSp(1/64) is a symmetry of M-theory. We also derive the effective action of the closed bosonic string as a non-linear realisation.

Keywords: Superstrings and Heterotic Strings, Supergravity Models.

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JHEP08(2000)007

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1. Introduction

Much of the progress in recent years in our understanding of the non-perturbative effects of string theory has relied on the structure of the supergravity theories in eleven and ten dimensions. While there is only one eleven-dimensional supergravity theory [1], in ten dimensions there exist the IIA [2, 3, 4] and IIB [5, 6, 7] supergravity theories as well as the type-I supergravity theory coupled to the Yang-Mills theory [8]. These theories are essentially uniquely determined by the type of supersymmetry that they possess. Hence given a string theory in ten dimensions its complete low energy effective action must be the supergravity with the same space-time supersymmetry. One intriguing feature of supergravity theories is the occurance of coset space symmetries that control the way the scalars in these theories behave. The four-dimensional $N = 4$ supergravity theory possess a $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ symmetry [9], the IIA theory a $\mathrm{SO}(1,1)$ symmetry [2], the IIB theory a $\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)$ [5] and the further reductions of the eleven-dimensional supergravity theory possess cosets based on the exceptional groups [10, 11]. These symmetries have also played an important role in string dualities in recent years [12, 13] and any further elucidation of the symmetries of supergravity theories could prove useful.

Although these symmetries can be viewed as a consequence of supersymmetry, it is desirable to have a deeper understanding. One step in this direction has been the extension of the coset space description of the scalars to include the gauge fields [14]. This has been achieved by introducing a group with Grassmann odd as well as Grassmann even generators. All of these generators are scalars under the Lorentz group and the indices of the gauge fields are incorporated by writing them as forms. The group elements g of the coset are then exponentials of these forms each of which is multiplied by a generator with the corresponding Grassmann character. This has the advantage that one automatically finds the gauge field strengths when taking the Cartan forms, $g^{-1}dg$ using the method of non-linear realisations [15, 16]. The result is an elegant formulation of these sectors of the supergravity theories, but it is not apparent how this method can be extended to include the graviton or indeed the fermionic sectors of the theory.

Recently, it was shown [17] that part of the GL(32) automorphism group of the supersymmetry algebra was found to be a symmetry of the fivebrane equations of motion. This symmetry was also found to play an important role in formulating the branes of M-theory in terms of a non-linear realisation, indeed the world-volume gauge field strengths are the Goldstone bosons for part of this automorphism symmetry [18]. It was conjectured [17] that this symmetry could play a role in M-theory and should occur in eleven-dimensional supergravity.

Long ago [19], Ogievetsky realised that the group of general coordinate realisations was the closure of the conformal group and the group of affine transformations, IGL(4) in four dimensions. As a consequence, in ref. [20], gravity was reformulated as the non-linear realisation of these two groups. Subsequently, it was shown [21] that the Sokatchev-Ogievetsky superspace formulation of the $N = 1$ supergravity [22] in four dimensions could be expressed as a non-linear realisation

In this paper we wish to revive this old idea of realising gravity as a non-linear realisation and, by combining it with the presence of the automorphism symmetries, show that the bosonic sector of eleven-dimensional supergravity can be expressed as a non-linear realisation. In particular, in section 2, we will show how the bosonic sector of sector of eleven-dimensional supergravity, that is the graviton and the rank three gauge field, is a non-linear realisation of the conformal group, $\mathrm{SO}(2,11)$ and a group which is generated by the generators of the group of affine transformations IGL(11) and two further generators which are of rank three and rank six. While the graviton is the Goldstone boson for the group GL(11), the gauge field and its dual are the Goldstone bosons associated with these two additional generators. We will argue that these new generators arise as part of the GL(32) automorphism group of the supersymmetry algebra in the fully supersymmetric theory. One puzzle with realising gauge fields as Goldstone fields is that one does not obviously find their field strengths when following the standard method of non-linear realisations. In fact, the field strengths of these gauge fields arise only as a result of demanding that the theory be invariant under both groups.

We will also show, in section 3, that the bosonic sector of IIA supergravity can also be derived as a non-linear realisation. In section 4, we show that if one starts with a generic theory of Goldstone fields, some of which carry anti-symmetrised space-time indices, and also demands conformal symmetry then one finds that these Goldstone fields must possess gauge symmetries. In section 5, we show that the low energy effective action for the closed bosonic string can be written as a non-linear realisation. We explain, in section 6, how one may derive the dynamics of branes in a background using the theory of non-linear realisations. We sketch, in section 7, how the non-linear realisation of the bosonic sectors of supergravity theories can be extended to the fully theory including the fermions. We conclude in section 8.

2. Eleven-dimensional supergravity

The lagrangian of eleven-dimensional supergravity written in the signature $\mathrm{diag}(\eta_{ab}) = (-1, 1, 1, \ldots, 1)$ is given by [1]

$$\begin{aligned} L = &+ \frac{e}{4\kappa^2} R(\Omega(e,\psi)) - \frac{e}{48} F_{\mu_1\cdots\mu_4} F^{\mu_1\cdots\mu_4} - \frac{e}{2}\bar{\psi}_\mu \Gamma^{\mu\nu\varrho} D_\nu \left(\frac{1}{2}(\hat{\hat{\Omega}} + \hat{\Omega})\right)\psi_\varrho - \\ &- \frac{1}{192} e\kappa (\bar{\psi}_{\mu_1}\Gamma^{\mu_1\cdots\mu_6}\psi_{\mu_2} + 12\bar{\psi}^{\mu_3}\Gamma^{\mu_4\mu_5}\psi^{\mu_6})(F_{\mu_3\cdots\mu_6} + \hat{F}_{\mu_3\cdots\mu_6}) + \\ &+ \frac{\kappa}{(12)^4}\epsilon^{\mu_1\cdots\mu_{11}} F_{\mu_1\cdots\mu_4} F_{\mu_5\cdots\mu_8} A_{\mu_9\mu_{10}\mu_{11}}\,, \end{aligned} \tag{2.1}$$

where

$$F_{\mu_1\cdots\mu_4} = 4\partial_{[\mu_1} A_{\mu_2\mu_3\mu_4]}\,, \qquad \hat{F}_{\mu_1\cdots\mu_4} = F_{\mu_1\cdots\mu_4} + 3\bar{\psi}_{[\mu_1}\Gamma_{\mu_2\mu_3}\psi_{\mu_4]}\,, \tag{2.2}$$

and

$$\hat{\hat{\Omega}}_{\mu bc} = \hat{\Omega}_{\mu bc} + \frac{1}{4}\bar{\psi}_\nu \Gamma_{\mu bc}{}^{\nu\lambda}\psi_\lambda\,, \qquad \hat{\Omega}_{\mu bc} = w_{\mu bc}(e) - \frac{1}{2}(\bar{\psi}_\nu\Gamma_c\psi_b - \bar{\psi}_\nu\Gamma_b\psi_c + \bar{\psi}_c\Gamma_\mu\psi_b)$$

and

$$\begin{aligned} w_{\mu bc}(e) = &\frac{1}{2}(e_b{}^\rho\partial_\mu e_{\rho c} - e_c{}^\rho\partial_\mu e_{\rho b}) - \frac{1}{2}(e_b{}^\rho\partial_\rho e_{\mu c} - e_c{}^\rho\partial_\rho e_{\mu b}) - \\ &- \frac{1}{2}(e_b{}^\lambda e_c{}^\rho\partial_\lambda e_{\rho a} - e_c{}^\lambda e_b{}^\rho\partial_\lambda e_{\rho a})e_\mu{}^a\,. \end{aligned} \tag{2.3}$$

The symbol $w^0_{\mu mn}(e)$ is the usual expression for the spin-connection in terms of the vielbein e^n_μ.

The equation of motion of the gauge field is given by

$$D_{\mu_1}F^{\mu_1\cdots\mu_4} + \frac{(\det e)^{-1}}{4.(12)^2}\epsilon^{\mu_2\cdots\mu_4\nu_1\cdots\nu_8}F_{\nu_1\nu_2\nu_3\nu_4}F_{\nu_5\nu_6\nu_7\nu_8} = 0\,. \tag{2.4}$$

This equation may be rewritten in first order form as

$$F_{\mu_1\cdots\mu_4} = \frac{\det e}{7!}\epsilon_{\mu_1\cdots\mu_4\nu_1\cdots\nu_7}F^{\nu_1\cdots\nu_7}\,, \tag{2.5}$$

where

$$\tilde{F}_{\nu_1\cdots\nu_7} \equiv 7\left(\partial_{[\nu_1}A_{\nu_2\cdots\nu_7]} + 5A_{[\nu_1\cdots\nu_3}F_{\nu_4\cdots\mu_7]}\right). \tag{2.6}$$

We consider the Lie algebra whose non-vanishing commutators are

$$\begin{aligned} [K^a{}_b, K^c{}_d] &= \delta^c_b K^a{}_d - \delta^a_d K^c{}_b\,, & [K^a{}_b, P_c] &= -\delta^a_c P_b\,, \\ [K^a{}_b, R^{c_1\cdots c_6}] &= \delta^{c_1}_b R^{ac_2\cdots c_6} + \cdots\,, & [K^a{}_b, R^{c_1\cdots c_3}] &= \delta^{c_1}_b R^{ac_2c_3} + \cdots\,, \end{aligned} \tag{2.7}$$

$$[R^{c_1\cdots c_3}, R^{c_4\cdots c_6}] = 2R^{c_1\cdots c_6}\,, \tag{2.8}$$

where $+\cdots$ denote the appropriate anti-symmetrisations. The generators $K^a{}_b$ and P_c generate the affine group IGL(11) while the generators $R^{c_1\cdots c_3}$ and $R^{c_1\cdots c_6}$ form a subalgebra that is the same as that which was found to be a symmetry of the fivebrane [17]. This subalgebra was also required in the description of the fivebrane as a non-linear realisation [18]. In these references it was identified as part of the GL(32) automorphism algebra of the eleven-dimensional supersymmetry algebra. We denote by G_{11} the group whose Lie algebra is that of eqs. (2.7) and (2.8).

We now construct the non-linear realisation corresponding to the group G_{11} taking the Lorentz group to be a local symmetry. The generators of the Lorentz group are given by $J_{ab} = K_{ab} - K_{ba}$ where the indices are lowered with the Minkowski metric. We therefore consider group elements of the form

$$g = e^{x^\mu P_\mu} e^{h_a{}^b K^a{}_b} \exp\left(\frac{A_{c_1\cdots c_3}R^{c_1\cdots c_3}}{3!} + \frac{A_{c_1\cdots c_6}R^{c_1\cdots c_6}}{6!}\right). \tag{2.9}$$

The fields $h^a{}_b$, $A_{c_1\cdots c_3}$ and $A_{c_1\cdots c_6}$ depend on x^μ. Although we use the exponential parameterization the reader who prefers a globally valid expression can readily rewrite the above group element in the appropriate form. In fact, one could take x^μ, $h^a{}_b$, $A_{c_1\cdots c_3}$ and $A_{c_1\cdots c_6}$ to depend on D parameters, thus leading to a kind of democracy between fields and coordinates. To recover the above form from this formulation, one uses the reparameterisation invariance inherent in the construction to choose the x^μ equal to the parameters.

The theory is to be invariant under

$$g \longrightarrow g_0 g h^{-1}\,, \tag{2.10}$$

where g_0 is a rigid element of the full group generated by the above Lie algebra and h is a local element of the Lorentz group. The corresponding g_0 invariant forms are given by

$$\mathcal{V} = g^{-1}dg - w\,, \tag{2.11}$$

where $w \equiv \frac{1}{2} dx^\mu w_{\mu b}{}^c J^b{}_c$ is the Lorentz connection and so transforms as

$$w \longrightarrow h w h^{-1} + h d h^{-1} . \tag{2.12}$$

As a result

$$\mathcal{V} \longrightarrow h \mathcal{V} h^{-1} . \tag{2.13}$$

This approach differs from that of ref. [20] where the Lorentz symmetry was a rigid symmetry and the field $h_a{}^b$ was symmetric. The advantage of the approach adopted here is that one finds directly the vielbein formulation of general relativity and so the identification of part of the theory with general relativity is readily apparent.

Evaluating $\mathcal{V}$ we find that

$$\begin{aligned} \mathcal{V} = dx^\mu \Bigg(& e_\mu{}^a P_a + \Omega_a{}^b K^a{}_b + \frac{1}{3!} \tilde{D}_\mu A_{c_1 \cdots c_3} R^{c_1 \cdots c_3} + \\ & + \frac{1}{6!} \tilde{D}_\mu A_{c_1 \cdots c_6} R^{c_1 \cdots c_6} \Bigg) , \end{aligned} \tag{2.14}$$

where

$$\begin{aligned} e_\mu{}^a &\equiv (e^h)_\mu{}^a , \qquad \tilde{D}_\mu A_{c_1 \cdots c_3} \equiv \partial_\mu A_{c_1 c_2 c_3} + \left(\left(e^{-1} \partial_\mu e \right)_{c_1}{}^b A_{b c_2 c_3} + \cdots \right) , \\ \tilde{D}_\mu A_{c_1 \cdots c_6} &\equiv \partial_\mu A_{c_1 \cdots c_6} + \left(\left(e^{-1} \partial_\mu e \right)_{c_1}{}^b A_{b c_2 \ldots c_6} + \cdots \right) - \left(A_{[c_1 \cdots c_3} \tilde{D}_\mu A_{c_4 \cdots c_6]} \right) , \\ w_{\mu b}{}^c &\equiv (e^{-1} \partial_\mu e)_b{}^c - w_{\mu b}{}^c , \end{aligned} \tag{2.15}$$

where $+\cdots$ denotes the action of $(e^{-1}\partial_\mu e)$ on the other indices of $A_{c_1 \cdots c_3}$ and $A_{c_1 \cdots c_6}$.

The covariant derivatives of the Goldstone fields associated with this non-linear realisation, that is of the field $h_a{}^b$ and the fields $A_{c_1 \cdots c_3}$ and $A_{c_1 \cdots c_6}$, are given by

$$\begin{aligned} \Omega_{ab}{}^c &\equiv (e^{-1})_a{}^\mu (e^{-1} \partial_\mu e)_b{}^c - w_{ab}{}^c , \\ \tilde{D}_a A_{c_1 \cdots c_3} &\equiv (e^{-1})_a{}^\mu \tilde{D}_\mu A_{c_1 \cdots c_3} , \qquad \tilde{D}_a A_{c_1 \cdots c_6} \equiv (e^{-1})_a{}^\mu \tilde{D}_\mu A_{c_1 \cdots c_6} . \end{aligned} \tag{2.16}$$

We note that these quantities are not field strengths as the indices are not anti-symmetrised, nor are the derivatives those that occur in general relativity. As we shall see, we will only recover the field strengths of the equations of motion once we consider the simultaneous non-linear realisation with the conformal group.

Under $h = e^{(\frac{1}{2} w_a{}^b(x) J^a{}_b)}$, we find that $e_\mu{}^a$ transforms as a elfbein should under a local Lorentz transformation and indeed all the indices of the fields in $\mathcal{V}$ which are contracted with the generators are rotated in the correct way as to be interpreted as tangent space indices. A matter field B transforms as $B \to B' = D(h) B$ where D is the representation of the Lorentz group to which B belongs. The covariant derivative of the matter field B is given by

$$\tilde{D}_a B \equiv (e^{-1})_a{}^\mu \partial_\mu B + \frac{1}{2} w_{ab}{}^c \Sigma^b{}_c B\,, \tag{2.17}$$

where $w_{ab}{}^c \equiv (e^{-1})_a{}^\mu w_{\mu b}{}^c$ and $\Sigma_a{}^b$ is the representation of the generators of the Lorentz group associated with B.

We must construct the non-linear realisation for the conformal group SO(2, 11), taking the now rigid Lorentz group as the istropy group. This procedure is well known, [20, 23], but for completeness we summarise the derivation. The generators of the conformal group obey the relations

$$\begin{aligned} [J_{ab}, K_c] &= -\eta_{ac} K_b + \eta_{bc} K_a\,, & [P_a, D] &= P_a\,, \\ [K_a, D] &= -K_a\,, & [P_a, K_b] &= +2\eta_{ab} D - 2J_{ab} \end{aligned} \tag{2.18}$$

in addition to those of the Poincare group and relations where the commutators vanish. We take as our coset representative

$$g = e^{x^\mu P_\mu} e^{\phi^\mu K_\mu} e^{\sigma D} \tag{2.19}$$

and the Cartan forms are given by

$$\begin{aligned} g^{-1} dg = dx^a (&e^\sigma P_a + e^{-\sigma} (\partial_a \phi^b - \phi^c \phi_c \delta_a^b + 2\phi_a \phi^b) K_b + (\partial_a \sigma + 2\phi_a) D + \\ &+ (-\delta_a^c \phi^d + \delta_a^d \phi^c) J_{cd})\,. \end{aligned} \tag{2.20}$$

The covariant derivatives of the Goldstone fields are obtained by taking all the above expressions, with the exception of the first term, and multiplying by $e^{-\sigma}$. These transform only under the Lorentz group and we can set the covariant derivative for σ to zero and still preserve the group. As a result, we can eliminate ϕ_μ in terms of $\partial_\mu \sigma$, namely $2\phi_\mu = -\partial_\mu \sigma$. In effect, this leaves σ as the only Goldstone field.

It is straightforward to find the transformations under dilations and special conformal transformations of σ and a field B which transforms under the representation Σ_{ab} of the Lorentz group. The result is

$$\begin{aligned} \delta\sigma &= (2x \cdot \beta x \cdot \partial - x^2 \beta \cdot \partial)\sigma + 2\beta_\mu x^\mu + \lambda\,, \\ \delta B &= (2x \cdot \beta x \cdot \partial - x^2 \beta \cdot \partial) B + (\beta^a x^b - \beta^b x^a) \Sigma_{ab} B\,. \end{aligned} \tag{2.21}$$

The covariant derivative with respect to conformal transformations, denoted Δ_a of the field B is given by

$$\Delta_a B = e^{-\sigma} (\partial_a + \partial^b \sigma \Sigma_{ab}) B\,. \tag{2.22}$$

In particular, for a vector field A_a we have

$$\Delta_a A_b = e^{-\sigma} (\partial_a A_b + \eta_{ab} \partial^c \sigma A_c - \partial_b \sigma A_a)\,. \tag{2.23}$$

Following the procedure of Borisov and Ogievetski [20], we must now construct quantities from the derivatives of the Goldstone fields of the first group G_{11} which are also covariant with respect to the conformal group. In view of the identical transformation of x^μ under dilations and the determinant part of GL(11) we must identify $h_\mu{}^a = \bar{h}_\mu^a + \sigma\delta_\mu^a$ where $\bar{h}_\mu{}^\mu = 0$. While the field σ identified in this way must transform in the relavent way determined by the conformal group, the fields $\bar{h}_\mu{}^a$, $A_{c_1\cdots c_3}$ and $A_{c_1\cdots c_6}$ transform under conformal transformations as their indices suggest. It is simplest to first carry out this procedure for the fields $A_{c_1\cdots c_3}$ and $A_{c_1\cdots c_6}$. The G_{11} covariant derivative of $A_{c_1\cdots c_3}$ can be rewritten as

$$\begin{aligned}\tilde{D}_a A_{c_1\cdots c_3} = (\bar{e})_a{}^\mu\Big(\Delta_\mu A_{c_1\cdots c_3} + e^{-\sigma}(-\eta_{\mu c_1}\partial^d\sigma A_{d\cdots c_3} + \\ + D_{c_1}\sigma A_{\mu\cdots c_3} + \partial_\mu\sigma A_{c_1\cdots c_3} + (\bar{e}^{-1}\partial_\mu\bar{e})_{c_1}{}^d A_{d\cdots c_3} + \cdots)\Big)\,.\end{aligned} \tag{2.24}$$

In this equation $\bar{e} = e^{\bar{h}}$ and $+\cdots$ denotes the terms that arise when the connections of the derivatives contract with the other indices on $A_{c_1\cdots c_3}$. Even at order $(\bar{h})^0$ it is apparent that only by completely anti-symmetrising in $a, c_1\cdots c_3$ can one obtain an expression such that all σ dependence is through the conformal derivative Δ_a alone and as a result is an expression that is simultaneously covariant under both groups. Thus the unique simultaneously covariant quantity is

$$\tilde{F}_{c_1\cdots c_4} \equiv 4\left(e_{[c_1}{}^\mu\partial_\mu A_{c_2\cdots c_4]} + e_{[c_1}{}^\mu(\bar{e}^{-1}\partial_\mu\bar{e})_{[c_2}{}^b A_{bc_3c_4]} + \cdots\right). \tag{2.25}$$

A similar calculation for the gauge field $A_{c_1\cdots c_6}$ leads to the unique simultaneously covariant expression

$$\tilde{F}_{c_1\cdots c_7} \equiv 7\left(e_{[c_1}{}^\mu\left(\partial_\mu A_{c_2\cdots c_7]}\right) + \Omega_{[c_1c_2}{}^b A_{bc_3c_7]} + \cdots + 5\tilde{F}_{[c_1\cdots c_4}\tilde{F}_{c_5\cdots c_7]}\right). \tag{2.26}$$

What is not apparent from the above expressions is that the covariant derivatives are those that one should find in general relativity. To verify this, and indeed to recover general relativity itself, we must recover the usual expression for the spin connection in terms of the elfbein. We can use the inverse Higgs effect [24] to place constraints on the covariant derivative $\Omega_{ab}{}^c$ of the field $h_\mu{}^a$. Within the context of the group G_{11} there is no unique way to do this. However, as explained above, we must do this in just such a way that the G_{11} covariant derivative of eq. (2.17) can be rewritten in terms of the covariant derivatives of the conformal group of eq. (2.21). Consider a matter field B as discussed above, its G_{11} covariant derivative can be expressed as

$$\tilde{D}_a B = (\bar{e})_a{}^\mu\left(\Delta_\mu B - e^{-\sigma}\partial_\nu\sigma\Sigma_\mu{}^\nu B + \frac{1}{2}e^{-\sigma}w_{\mu b}{}^c\Sigma^b{}_c B\right). \tag{2.27}$$

This equation tells us that $w_{ab}{}^c$ must be expressible as in terms of the conformal covariant derivatives of $\bar{e}_\mu{}^a$ as well as a derivatives of σ which must cancel the second term in the right-hand side of the above equation. The unique solution is to take the constraint

$$\Omega_{a[bc]} - \Omega_{b(ac)} + \Omega_{c(ab)} = 0\,. \tag{2.28}$$

This results in the well known expression for the spin connection in terms of the elfbein given in eq. (2.3). Although the connection term that appears in the field strengths of eqs. (2.25) and (2.26) looks incorrect, when one takes into account the anti-symmetry on all the indices one finds that the covariant derivative can be re-expressed in terms of the usual spin connection appropriate to tangent indices. Thus the field strengths of these equations when written in terms of the components appropriate for the coordinates of the space-time are just the curl of the gauge potential written in the same components.

The equations of motion for the simultaneous non-linear realisation must be written in terms of $\tilde{F}_{c_1\cdots c_4}$ and $\tilde{F}_{c_1\cdots c_7}$ given in eqs. (2.25) and (2.26) and the spin-connection in such a way that the equations are covariant under the local Lorentz transformations. Clearly, the spin connection can only enter either in $\tilde{F}_{c_1\cdots c_4}$ and $\tilde{F}_{c_1\cdots c_7}$, in the way which is already specified, or through the Riemann tensor

$$R_{\mu\nu b}{}^{c} \equiv \partial_\mu w_{\nu b}{}^{c} + w_{\mu b}{}^{d} w_{\nu d}{}^{c} - (\mu \to \nu)\,. \tag{2.29}$$

The unique first order equation for the gauge field which is not trivial is

$$\tilde{F}_{c_1\cdots c_4} = \frac{1}{7!}\epsilon_{c_1\cdots c_{11}}\tilde{F}^{c_5\cdots c_{11}} \tag{2.30}$$

in agreement with eq. (2.5) when written in the local coordinate frame. The only other non-trivial equation is

$$R_{\mu\nu b}{}^{c} e_c{}^{\nu} e_a{}^{\mu} - \frac{1}{2}\eta_{ab} R_{\mu\nu b}{}^{c} e_c{}^{\nu} e^{b\mu} - \frac{c}{4}\left(\tilde{F}_{ac_1\cdots c_3}\tilde{F}_b{}^{c_1\cdots c_3} - \frac{1}{6}\eta_{ab}\tilde{F}_{c_1\cdots c_4}\tilde{F}^{c_1\cdots c_4}\right) = 0 \tag{2.31}$$

as it should be. The constant c, of proportionality can only be determined once the full supersymmetric treatment is given. It has value 1.

3. Gauge symmetry

It is instructive to trace more precisely how the gauge invariance of the fields $A_{a_1\cdots a_3}$ and $A_{a_1\cdots a_6}$ arises as a consequence of the simultaneous realisation of G_{11} and the conformal group. Taking

$$g_0 = \exp\left(\frac{c_{\mu_1\cdots\mu_3}\delta^{\mu_1\cdots\mu_3}_{a_1\cdots a_3}R^{a_1\cdots a_3}}{3!} + \frac{c_{\mu_1\cdots\mu_6}\delta^{\mu_1\cdots\mu_6}_{a_1\cdots a_6}R^{a_1\cdots a_6}}{6!}\right), \tag{3.1}$$

where $\delta^{\mu_1\cdots\mu_n}_{a_1\cdots a_n} = \delta^{a_1}_{\mu_1}\cdots\delta^{a_n}_{\mu_n}$ and $c_{\mu_1\cdots\mu_3}$ and $c_{\mu_1\cdots\mu_6}$ are constants in eq. (2.10), we find that the vielbein is inert and the other fields transform as

$$\delta A_{a_1\cdots a_3} = c_{a_1\cdots a_3}\,, \qquad \delta A_{a_1\cdots a_6} = c_{a_1\cdots a_6} + 20 c_{[a_1\cdots a_3}A_{a_4\cdots a_6]}\,, \tag{3.2}$$

where $c_{\mu_1\cdots\mu_3} = e_{\mu_1}{}^{a_1}\cdots e_{\mu_3}{}^{a_3}c_{a_1\cdots a_3}$ and similarly for $c_{a_1\cdots a_6}$. The vielbeins occur because the factor in g which contains the fields $A_{a_1\cdots a_3}$ and $A_{a_1\cdots a_6}$ is to the right of that containing the $h_a{}^b$ fields. Thus, it is the fields with curved indices that transform most simply. To find the conformal transformation of the fields with curved indices we write them as $A_{\mu_1\cdots\mu_p} = (e^{\bar{h}})_{\mu_1}{}^{a_1}\cdots(e^{\bar{h}})_{\mu_p}{}^{a_p}e^{p\sigma}A_{a_1\cdots a_p}$ for $p=3,6$. Taking into account the conformal transformation of σ and B of eq. (2.21) we find that

$$\begin{aligned}\delta A_{\mu_1\cdots\mu_p} &= (2(x\cdot\beta)(x\cdot\partial) - x^2(\beta\cdot\partial))A_{\mu_1\cdots\mu_p} + \\ &\quad + (2\beta_{\mu_1}x^\kappa A_{\kappa\mu_2\cdots\mu_p} - 2x_{\mu_1}\beta^\kappa A_{\kappa\mu_2\cdots\mu_p} + \cdots) + 2p(\beta\cdot x)A_{\mu_1\cdots\mu_p}\,,\end{aligned} \tag{3.3}$$

where $+\cdots$ stands for the other terms where the induced Lorentz rotation acts on the other indices of $A_{\mu_1\cdots\mu_p}$. We note that this is the variation of a matter field that we would have obtain had we included the dilations in the isotropy group and assigned the field dilation weight p.

For simplicity, we will illustrate the mechanism of how gauge symmetry arises for a single form field $A_{\mu_1\cdots\mu_p}$ that has a constant shift, i.e. $\delta A_{\mu_1\cdots\mu_p} = c_{\mu_1\cdots\mu_p}$, under a non-linear realisation. Carrying out the commutation of this shift with a special conformal transformation we find that

$$[\delta_c, \delta_\beta]A_{\mu_1\cdots\mu_p} = p\partial_{[\mu_1}\tilde{\Lambda}^{(2)}_{\mu_2\cdots\mu_p]}\,. \tag{3.4}$$

We recognise this as a gauge transformation with parameter

$$\tilde{\Lambda}^{(2)}_{\mu_2\cdots\mu_p} = 2px\cdot\beta x^\kappa c_{\kappa\mu_2\cdots\mu_p} - x^2\beta^\kappa c_{\kappa\mu_2\cdots\mu_p} + (-2x_{\mu_2}\beta^\kappa x^\rho c_{\rho\kappa\mu_3\cdots\mu_p} + \cdots)\,. \tag{3.5}$$

We may write the original shift of the field as a gauge transformation with parameter $\Lambda^{(1)}_{\mu_2\cdots\mu_p} = x^\kappa c_{\kappa\mu_2\cdots\mu_p}$ and taking its commutator with special conformal transformations we find another gauge transformation which is quadratic in x^μ. Thus starting from a gauge transformation that is linear in x^μ we obtain one which is bilinear.

By induction, we will now show that taking repeated commutators with special conformal transformations leads to a gauge transformation with an arbitrary local parameter. Let us suppose that after taking $r-1$ commutators we find a transformation which can be written as a gauge transformation of $A_{\mu_1\cdots\mu_p}$, denoted $\Lambda^{(r)}_{\mu_2\cdots\mu_p}$, which is a polynomial in x^μ of degree r. Taking the commutator of this with another special conformal transformation we find that

$$[\delta_\Lambda, \delta_\beta]A_{\mu_1\cdots\mu_p} = p\partial_{[\mu_1}\tilde{\Lambda}^{(r+1)}_{\mu_2\cdots\mu_p]}\,, \tag{3.6}$$

where

$$\begin{aligned}\tilde{\Lambda}^{(r+1)}_{\mu_2\cdots\mu_p]} &= (2x\cdot\beta x\cdot\partial - x^2\beta\cdot\partial)\Lambda^{(r)}_{\mu_2\cdots\mu_p} + \left(2\beta_{\mu_2}x^\kappa\Lambda^{(r)}_{\kappa\mu_3\cdots\mu_p} - 2x_{\mu_2}\beta^\kappa\Lambda^{(r)}_{\kappa\mu_3\cdots\mu_p} + \cdots\right) \\ &\quad + 2(p-1)x\cdot\beta\Lambda^{(r)}_{\mu_2\cdots\mu_p}\,.\end{aligned} \tag{3.7}$$

Hence we recover another gauge transformation, which is a polynomial in x^μ of one degree higher. It is clear that proceeding in this way we can find an arbitrary local gauge transformation. Thus, we have shown that taking the closure of a Goldstone shift and the conformal group leads to a local gauge transformation. In fact, if we start from a non-linear realisation of the fields we can regard gauge invariance as a consequence of conformal invariance.

The standard U(1) field has been been previously considered as a Goldstone boson by considering an infinite dimensional algebra [29].

4. IIA supergravity

The bosonic part of the ten-dimensional IIA supergravity theory is given by [2, 3, 4]

$$\begin{aligned} L^B = & \, eR(w(e)) - \frac{1}{12} e e^{\sigma/2} F'_{\mu_1\cdots\mu_4} F'^{\mu_1\cdots\mu_4} - \frac{1}{3} e e^{-\sigma} F_{\mu_1\cdots\mu_3} F^{\mu_1\cdots\mu_3} - \\ & - e e^{\frac{3}{2}\sigma} F_{\mu_1\mu_2} F^{\mu_1\mu_2} - \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \\ & + \frac{1}{2\cdot(12)^2}\epsilon^{\mu_1\cdots\mu_{10}} F_{\mu_1\cdots\mu_4} F_{\mu_5\cdots\mu_8} A_{\mu_9\mu_{10}} \,, \end{aligned} \tag{4.1}$$

where

$$F_{\mu_1\mu_2} = 2\partial_{[\mu_1} A_{\mu_2]} \,, \tag{4.2}$$
$$F_{\mu_1\mu_2\mu_3} = 3\partial_{[\mu_1} A_{\mu_2\mu_3]} \,, \tag{4.3}$$
$$F'_{\mu_1\cdots\mu_4} = 4(\partial_{[\mu_1} A_{\cdots\mu_4]} + 2A_{[\mu_1} F_{\mu_2\mu_3\mu_4]}) \,. \tag{4.4}$$

The equations of motion of the non-gravitational sector of the theory were expressed as a non-linear realisation in ref. [14]. Although we use a different group and strategy, some of the steps below have analogues in those of ref. [14]. We proceed much as for eleven-dimensional supergravity, we take the group $G_{\rm IIA}$ to have the generators of IGL(10), that obey eq. (2.17), together with the generators $R_{a_1\cdots a_p}$ for $p = 0, 1, 2, 3, 5, 6, 7, 8$ which obey the relations

$$[R, R_{a_1\cdots a_p}] = c_p R_{a_1\cdots a_p} \,, \qquad [R_{a_1\cdots a_p}, R_{a_1\cdots a_q}] = c_{p,q} R_{a_1\cdots a_{(p+q)}} \,, \tag{4.5}$$

where

$$\begin{aligned} & c_1 = -c_7 = -\frac{3}{4} \,, \qquad c_2 = -c_6 = \frac{1}{2} \,, \qquad c_3 = -c_5 = -\frac{1}{4} : \\ & c_{1,2} = -c_{2,3} = -c_{3,3} = c_{2,5} = c_{1,5} = 2 \,, \\ & c_{1,7} = 3 \,, \qquad c_{2,6} = 2 \,, \qquad c_{3,5} = 1 \end{aligned} \tag{4.6}$$

and all other c's vanish. We take the group element of $G_{\rm IIA}$ to be of the form

$$g = e^{x^\mu P_\mu} g_h g_A \,, \tag{4.6}$$

where

$$g_h = e^{h_a{}^b K^a{}_b}\,,$$

and

$$g_A = \exp\left(\frac{A_{a_1\cdots a_8}}{8!}R^{a_1\cdots a_8}\right)\cdots\exp\left(\frac{A_{a_1\cdots a_3}}{3!}R^{a_1 a_2 a_3}\right)\times \\ \times\exp\left(\frac{A_{a_1 a_2}}{2!}R^{a_1 a_2}\right)\exp(A_a R^a)\exp(AR)\,. \tag{4.7}$$

We also take the Lorentz group to be a local symmetry and so consider the quantity

$$\mathcal{V} = g^{-1}dg - w\,. \tag{4.8}$$

We may rewrite this as

$$\begin{aligned}\mathcal{V} &= \left(g_h^{-1}dg_h\right) + \left(g_A^{-1}dg_A + g_A^{-1}(g_h^{-1}dg_h)g_A - g_h^{-1}dg_h\right)\\ &\equiv dx^\mu(e_\mu{}^a P_a + \Omega_{\mu a}{}^b K^a{}_b) + dx^\mu\left(\sum_{p=1}^{8}\frac{1}{p!}\tilde{D}_\mu A_{a_1\cdots a_p}R^{a_1\cdots a_p}\right),\end{aligned} \tag{4.9}$$

where the definition applies to each of the terms in the brackets separately.

The next step is to demand conformal invariance. In particular, we should take combinations of derivatives of the Goldstone fields that are also conformally covariant. The procedure follows closely those of sections 2 and 3. Indeed, it is an inevitable consequence of section 3 that the fields $A_{a_1\cdots a_p}$ will appear in quantities that are gauge invariant. Thus the quantities which involve the fields $A_{a_1\cdots a_p}$ that are G_{IIA} and conformally covariant are

$$\tilde{F}_{a_1\cdots a_p} \equiv pe^{-c_{(p-1)}A}\tilde{D}_{[a_1}A_{a_2\cdots a_3]}\,. \tag{4.10}$$

One finds that

$$\begin{aligned}\tilde{F}_a &= D_a A\,, \qquad \tilde{F}_{a_1 a_2} = 2e^{\frac{3}{4}A}D_{[a_1}A_{a_2]}\,, \qquad \tilde{F}_{a_1 a_2 a_3} = 3e^{-\frac{1}{2}A}D_{[a_1}A_{a_2 a_3]}\,,\\ \tilde{F}_{a_1 a_2 a_3 a_4} &= 4e^{\frac{1}{4}A}\left(D_{[a_1}A_{a_2 a_3 a_4]} + 2e^{\frac{1}{2}A}A_{[a_1}\tilde{F}_{a_2 a_3 a_4]}\right), \qquad \tilde{F}_{a_1 a_2 a_3 a_4 a_5} = 0\,,\\ \tilde{F}_{a_1\cdots a_6} &= 6e^{-\frac{1}{4}A}\left(D_{[a_1}A_{a_2\cdots a_6]} + 5e^{-\frac{1}{4}A}A_{[a_1 a_2}\tilde{F}_{a_3\cdots a_6]}\right),\\ \tilde{F}_{a_1\cdots a_7} &= 7e^{\frac{1}{2}A}\left(D_{[a_1}A_{a_2\cdots a_7]} - 5e^{-\frac{1}{4}A}A_{[a_1 a_2 a_3}\tilde{F}_{a_4\cdots a_7]} + 2e^{\frac{1}{4}A}A_{[a_1}\tilde{F}_{a_2\cdots a_7]}\right),\\ \tilde{F}_{a_1\cdots a_8} &= 8e^{-\frac{3}{4}A}\left(D_{[a_1}A_{a_2\cdots a_8]} - 7e^{\frac{1}{4}A}A_{[a_1 a_2}\tilde{F}_{a_3\cdots a_8]}\right),\\ \tilde{F}_{a_1\cdots a_9} &= 9\Bigg(D_{[a_1}A_{a_2\cdots a_9]} - 8e^{-\frac{1}{2}A}A_{[a_1 a_2}\tilde{F}_{a_3\cdots a_9]} + \frac{7.4}{3}e^{\frac{1}{4}A}A_{[a_1 a_2 a_3}\tilde{F}_{a_4\cdots a_9]} +\\ &\qquad + 3e^{\frac{3}{4}A}A_{[a_1}\tilde{F}_{a_2\cdots a_9]}\Bigg)\,.\end{aligned} \tag{4.11}$$

The unique equations of motion of the forms that are first order in derivatives and constructed from the simultaneously covariant derivatives of the Goldstone fields are

$$\tilde{F}^{a_1\cdots a_p} = \frac{1}{(10-p)!}\epsilon^{a_1\cdots a_{10}}\tilde{F}_{a_{(p+1)}\cdots a_{10}}\,, \quad p = 1, 2, 3, 4 \tag{4.12}$$

as well as the equation for the vielbein. These are the equations of motion of IIA supergravity.

One might have thought that the easiest way to obtain the non-linear realisation of IIA supergravity would be to directly perform the dimensional reduction on the non-linear realisation of eleven-dimensional supergravity. However, the correct number of fields and generators does not arise in a natural way as becomes apparent if one tries. This suggests that the formulation of eleven-dimensional supergravity given in section 2 may not be the most natural one and that there should exist a first order formulation of the vielbein equation of motion by introducing higher rank fields.

5. The closed bosonic string effective action

One can also apply the theory of non-linear realisations to the low energy effective action of the closed string. This has been found to be [26]

$$\int d^D x \det e \left(R - \frac{4}{(D-2)}\partial_\mu\phi\partial^\mu\phi - \frac{1}{3}e^{-\frac{8\phi}{D-2}}F_{\mu_1\mu_2\mu_3}F^{\mu_1\mu_2\mu_3} \right), \tag{5.1}$$

where $F_{\mu_1\mu_2\mu_3} = 3\partial_{[\mu_1}A_{\mu_2\mu_3]}$. In this action, D is the dimension of space-time, but we must take $D = 26$ to obtain the consistent closed bosonic string. In principle we should include the cosmological term for $D \neq 26$ and there will also be corrections to the dilaton potential from higher order genus surfaces [26]. Presumeably, these terms could be accounted for by taking into account an anomaly in the symmetry below.

We then consider the group G_D whose generators are $K^a{}_b$, R, $R^{a_1a_2}$, $R^{a_1\cdots a_{(D-4)}}$ and $R^{a_1\cdots a_{(D-4)}}$. They obey eq. (2.7) and the algebra

$$[R, R^{a_1\cdots a_p}] = c_p R^{a_1\cdots a_p}\,, \qquad [R^{a_1\cdots a_p}, R^{a_1\cdots a_q}] = c_{p,q}R^{a_1\cdots a_{(p+q)}}\,, \tag{5.2}$$

where

$$c_2 = -c_{D-4} = \frac{4}{(D-2)}\,, \qquad c_{2,D-4} = 2\,. \tag{5.3}$$

We take and all the other c's vanish.

The non-linear realisation of G_D is built out of the group element $g = g_h g_A$ where

$$g_A = \exp\left(\frac{A_{a_1\cdots a_{(D-2)}}R^{a_1\cdots a_{(D-2)}}}{(D-2)!}\right)\exp\left(\frac{A_{a_1\cdots a_{(D-4)}}R^{a_1\cdots a_{(D-4)}}}{(D-4)!}\right)\times$$
$$\times \exp\left(\frac{A_{a_1a_2}R^{a_1a_2}}{(2)!}\right)\exp(AR)\,. \tag{5.4}$$

Calculating the Cartan forms $g^{-1}dg - w$ and demanding simultaneous invariance under the conformal group, we find that the equations of motion must be built out of $w_{ab}^{\ c}$ and

$$\begin{aligned}\tilde{F}_a &= D_a A\,, \qquad \tilde{F}_{a_1a_2a_3} = 3e^{-\frac{4}{(D-2)}A} D_{[a_1}A_{a_2a_3]}\,,\\ \tilde{F}_{a_1\cdots a_{(D-3)}} &= (D-3)e^{\frac{4}{(D-2)}A} D_{[a_1}A_{a_2\cdots a_{(D-3)}]}\,,\\ \tilde{F}_{a_1\cdots a_{(D-1)}} &= (D-1)\left(D_{[a_1}A_{a_2\cdots a_{(D-1)}]} + (D-2)e^{-\frac{4}{(D-2)}A}A_{[a_1a_2}F_{a_3\cdots a_{(D-1)}]}\right). \end{aligned} \tag{5.5}$$

The equations of motion are given by

$$\tilde{F}^{a_1\cdots a_p} = \frac{1}{(D-p)!}\epsilon^{a_1\cdots a_D}\tilde{F}_{a_{(p+1)}\cdots a_D}\,, \quad p = 1, 2 \tag{5.6}$$

provided we identify ϕ with A, as well as the vielbein equation.

6. Branes in a background

In a recent paper [18], the branes of M-theory were derived as a non-linear realisation. Since in this paper we have shown that the background supergravity to which they couple can also be formulated as a non-linear realisation, it is straightforward, at least in principle, to describe the dynamics of branes in a background as a non-linear realisation. We now illustrate the procedure for the case of a bosonic brane coupled to gravity.

We begin with a group element of $\mathrm{IGL}(D)$ of the form

$$g = e^{X^{\underline{a}}(\xi)P_{\underline{a}}}e^{h_{\underline{a}}{}^{\underline{b}}(X)K^{\underline{a}}{}_{\underline{b}}}\,. \tag{6.1}$$

We use the same index notation as in ref. [18], where D is the dimension of space-time and ξ^n are the coordinates of the brane worldvolume. We consider the forms

$$\mathcal{V} = g^{-1}dg - w \equiv d\xi^n(e_n{}^aP_a + f_n{}^{a'}P_{a'} + \omega_{n\underline{a}}{}^{\underline{b}}K^{\underline{a}}{}_{\underline{b}})\,. \tag{6.2}$$

The spin connection w takes values in the Lie algebra of $\mathrm{SO}(1,p)\times\mathrm{SO}(D-p-1)$.

In fact, in ref. [18], we took the isotropy group to be $\mathrm{SO}(1,p)$, although we could have taken the above group. Making this latter choice simplifies the analysis of ref. [18] a bit, but the results are the same.

Returning to the bosonic brane in a background, we find that

$$e_n{}^a = \partial_n X^{\underline{m}}e_{\underline{m}}{}^a\,, \qquad f_n{}^{a'} = \partial_n X^{\underline{m}}e_{\underline{m}}{}^{a'}\,. \tag{6.3}$$

Since $f_n^{a'}$ transforms in a covariant manner, we can set it to zero. This solves for $\partial_n X^{a'}$ in terms of the $e_n{}^{a'}$ of the background gravity. Hence, in the case of a local background we find that the parts of the vielbein that belong to the coset $\frac{\mathrm{SO}(1,D-1)}{\mathrm{SO}(1,p)\times\mathrm{SO}(D-p-1)}$

play the role of the Goldstone bosons of the Lorentz group that were solved for in ref. [18]. Proceeding as in that paper, and using the constraint ${f_n}^{a'} = 0$, we find that

$$ {e_n}^a \eta_{ab} {e_m}^b = \partial_n X^{\underline{p}} g_{\underline{pq}} \partial_m X^{\underline{q}} , \tag{6.4} $$

where $g_{\underline{nm}} = e_{\underline{m}}{}^{\underline{a}} \eta_{\underline{ab}} e_{\underline{n}}{}^{\underline{b}}$. Thus, we find that the invariant action is

$$ \int d^p\xi \det e^a_n = \int d^p\xi \sqrt{-\det(\partial_n X^{\underline{p}} g_{\underline{pq}} \partial_m X^{\underline{q}})} \tag{6.5} $$

as it should be.

7. Supersymmetric extension

It would be interesting to extend the analysis of this paper to the full supergravity theories, that is incorporate supersymmetry. Let us first sketch how this would go for eleven-dimensional supergravity. To extend the group IGL(11), it is natural to consider the group IGL(11/32). The generators of GL(11/32) group can be labelled by ${K^A}_B$ where $A = (a, \alpha)$ and similarly for B etc. We can then denote the generators by ${K^A}_B = ({K^a}_b, {K^a}_\alpha, {K^\alpha}_a, {K^\alpha}_\beta)$ and the generators of inhomogeneous transformations by P_a, K_α. The non-linear realisation is then built from the group elements of the form

$$ g = e^{(X^a P_a + K^\alpha \theta_\alpha)} e^{({h_a}^b {K^a}_b + {A_\alpha}^\beta {K^\alpha}_\beta)} e^{({\psi_a}^\alpha {K^a}_\alpha + {\zeta_\alpha}^a {K^\alpha}_a)} . \tag{7.1} $$

In the group element of eq. (7.1), the fields ${h_a}^b$, ${A_\alpha}^\beta$, ${\psi_a}^\alpha$ and ${\zeta_\alpha}^a$ are functions of x^a and θ_α and have geometric dimensions 0, 0, 1/2 and $-1/2$, respectively. Thus the lowest components of ${h_a}^b$, ${A_\alpha}^\beta$ and ${\psi_a}^\alpha$ have the correct dimensions to be identified with the graviton, gauge fields and gravitino, respectively. Indeed, their shifts under the appropriate symmetries of the non-linear realisations make this identification inevitable.

We must also consider a non-linear realisation of a supersymmetric generalisation of the conformal group. It is known [25], that there is a unique generalisation of SO(2, 11) that also contains the supersymmetry algebra: it is OSp(1/64). The Lie algebra of this group can be written as

$$ \begin{aligned} [R_{\hat\alpha\hat\beta}, R_{\hat\gamma\hat\delta}] &= -C_{\hat\beta\hat\gamma} R_{\hat\alpha\hat\delta} - C_{\hat\alpha\hat\delta} R_{\hat\gamma\hat\beta} - C_{\hat\beta\hat\delta} R_{\hat\alpha\hat\gamma} - C_{\hat\alpha\hat\gamma} R_{\hat\beta\hat\delta} , \\ \{\rho_{\hat\alpha}, \rho_{\hat\beta}\} &= R_{\hat\alpha\hat\beta} , \qquad [\rho_{\hat\gamma}, R_{\hat\alpha\hat\beta}] = C_{\hat\beta\hat\gamma} \rho_{\hat\alpha} + C_{\hat\alpha\hat\gamma} \rho_{\hat\beta} , \end{aligned} \tag{7.2} $$

where $C_{\hat\beta\hat\gamma} = -C_{\hat\gamma\hat\beta}$ is the metric that occurs in the invariant line element of this group and $\hat\alpha, \hat\beta = 1, 2, \ldots, 64$.

We now decompose the 64 component spinor, $\rho_{\hat\alpha}$ in this group into two 32 component spinors using the index decomposition $\hat\alpha = (\alpha, \alpha')$ where $\alpha = 1, \ldots, 32, \alpha' = 1, \ldots, 32$ etc. In particular, we set $Q_\alpha = \rho_\alpha$ and $S^\alpha = \rho_{\beta'} C^{\beta'\alpha}$, $R_{\alpha\beta} = Z_{\alpha\beta}$,

$R_\alpha{}^\beta = R_{\alpha\beta'}C^{\beta'\beta}$ and $Z^{\alpha\beta} = R_{\alpha'\beta'}C^{\alpha'\alpha}C^{\beta'\beta}$. Taking $C_{\alpha\beta} = 0 = C_{\alpha'\beta'}$ we may write the algebra of OSp(1/64) in the form

$$\begin{aligned} &\{Q_\alpha, Q_\beta\} = Z_{\alpha\beta}\,, \qquad [Q_\alpha, Z_{\gamma\delta}] = 0\,, \qquad [Z_{\alpha\delta}, Z_{\gamma\beta}] = 0\,, \\ &[Q_\alpha, R_\gamma{}^\delta] = -\delta_\alpha^\delta Q_\gamma\,, \qquad [Z_{\alpha\beta}, R_\gamma{}^\delta] = -\delta_\alpha^\delta Z_{\gamma\beta} - \delta_\beta^\delta Z_{\gamma\alpha} \end{aligned} \tag{7.3}$$

and

$$\begin{aligned} &\{S^\alpha, S^\beta\} = Z^{\alpha\beta}, \qquad [S^\alpha, Z^{\gamma\delta}] = 0\,, \qquad [Z^{\alpha\delta}, Z^{\gamma\beta}] = 0\,, \\ &[S^\gamma, R_\alpha{}^\beta] = \delta_\alpha^\gamma S^\beta\,, \qquad [Z^{\alpha\beta}, R^\gamma{}_\delta] = \delta_\gamma^\beta Z^{\alpha\delta} + \delta_\gamma^\alpha Z^{\delta\beta} \end{aligned} \tag{7.4}$$

as well as

$$\{Q_\alpha, S^\beta\} = R_\alpha{}^\beta\,, \qquad [Z_{\alpha\beta}, Z^{\gamma\delta}] = -\delta_\beta^\gamma R_\alpha{}^\delta - \delta_\alpha^\delta R_\beta{}^\gamma - \delta_\beta^\delta R_\alpha{}^\gamma - \delta_\alpha^\gamma R_\beta{}^\delta\,. \tag{7.5}$$

We recognise that OSp(1/64) contains a sub-algebra, given in eq. (7.3), which is precisely the usual supersymmetry algebra in eleven dimensions with all its central charges, plus the GL(32) automorphism group that was found to play a role in the fivebrane equations of motion [17] and in the branes of M-theory realised as a non-linear realisation [18]. Expanding in γ-matrices, we can express

$$R^\alpha{}_\beta = \sum_n R^{a_1\cdots a_n}(\gamma_{a_1\cdots a_n})^\alpha{}_\beta\,. \tag{7.6}$$

The generators R and $R_{a_1a_2}$ are to be identified with dilations and Lorentz rotations. The considerations of this paper show that OSp(1/64) must be a symmetry of eleven dimensional supergravity. This group has previously been considered [30] in the context of M theory with two times and mentioned as a possible unifying group in reference [31].

When taking the simultaneous realisation of the two groups IGL(11/32) and OSp(1/32) we must identify the Goldstone fields whose corresponding generators have the same action on the coordinates x^a and θ^α. In principle, one should also consider the action of OSp(1/64) on the central charges, but for the present discussion we shall ignore this subtilty. As for the bosonic sector consider in section 2, the dilations are in common and so we must identify $h^\mu{}_\mu$ with σ, However, the generators $K^\alpha{}_\beta$ and $R_\alpha{}^\beta$ act the same way on the coordinates x^a and θ^α with the exception of the scalar and rank two generators that behave differently. In OSp(1/64) these are the dilations and Lorentz rotations and so their actions on the coordinates x^a and θ^α are related. In contrast, the GL(11/32) action on the coordinates x^a and θ^α is unrelated. Thus, we should identify all of the generators in $K^\alpha{}_\beta$ with those in $R_\alpha{}^\beta$ with the exception of the two generators of rank zero and two. As a result, in the simultaneous non-linear realisation we find that we have generators of every rank in $R_\alpha{}^\beta$, including the dilations and Lorentz rotations, as well as two additional generators of rank zero and two, which we may denote by K and K_{ab}. Hence, in eleven dimensions, we find the automorphisms $R^{a_1\cdots a_n}$ for $n = 0, 1, 2, 3, 4, 6$ and the two-additional generators, K and $K_{a_1a_2}$. It was observed on ref. [17] that the algebra of eq. (2.8) was a contraction of the GL(32) automorphism algebra. Hence, it would seem natural to identify the generators $R^{a_1\cdots a_3}$ and $R^{a_1\cdots a_6}$, which are the generators, whose Goldstone fields are the gauge fields of eleven-dimensional supergravity in the non-linear realisation of section 2, with the automorphisms that arise in the groups GL(11/32) and OSp(1/64).

The supergravity action of eq. (2.1) essentially contains three contributions, the kinetic terms in the first line, the Noether term in the second line and the Chern-Simmons term in the last line. We have already accounted for the first and last terms and in the supersymmetric extension we must account for the Noether term. In the Cartan forms of the group element of eq. (7.1) we find a term of the form

$$e^{-\psi_a{}^\alpha K^a{}_\alpha}(\tilde{D}_a A_{a_1\cdots a_3} R^{a_1\cdots a_3} + \tilde{D}_a A_{a_1\cdots a_6} R^{a_1\cdots a_6}) e^{\psi_a{}^\alpha K^a{}_\alpha} \,. \tag{7.7}$$

Taking the commutator of the $R^{a_1\cdots a_3}$ and $R^{a_1\cdots a_6}$ generators with $K^a{}_\alpha$ to be the obvious γ-matrix times $K^a{}_\alpha$ we do indeed find a term that has, at least in form, that of the Noether term.

We now also briefly, comment on the supersymmetric extension of the IIA theory considered in section 4. The extension of the group IGL(10) is presumably the group IGL(10/32). Although ref. [25] was concerned with $N = 1$ supersymmetry, it would seem inevitable that the unique extension of the conformal group to include a type-II superalgebra is the group OSp(1/64). We should consider the simultaneous realisation of both of these groups. In the later group, we will find the GL(32) automorphism group and so the generators $R^{a_1\cdots a_p}$ for $p = 0, 1\cdots 10$. Identifying the generators in the same way as above we find that the simultaneous non-linear realisation of these two groups includes the generators $R^{a_1\cdots a_n}$ for $n = 0, 1, 2, \ldots, 10$ as well as two additional generators, which we can denote by K and $K_{a_1a_2}$. We note that in contrast to the eleven-dimensional theory most of the generators are need to ensure the necessary Goldstone bosons. The correspondence with the automorphisms of the supersymmetry algebra is less obvious in this case and it is possible that one may have to introduce generators in addition to those of the above two groups. In particular, the generator R, which leads to the SO$(1,1)$ transformations of the IIA supergravity theory, does not seem to have an obvious identification with these generators.

The non-linear realisation of the IIb supergravity theory follows a similar pattern, but the automorphisms that are active are different from those in the eleven-dimensional and IIA supergravity theories.

The above is a sketch of the extension to the supergravity theory, however, until one actually carries out the full calculation one cannot be sure that all the considerations in this section are correct.

8. Conclusion

It is clear that all supergravity theories can be formulated as non-linear realisations. The bosonic part of the group underlying these constructions will include, the conformal group, the general linear group and certain automorphisms. In the complete theory, the conformal group will be embedded in the relevant Osp group which automatically contain the automorphisms of the Poincare supersymmetry algebras with all their central charges.

For many years it has been a puzzle to understand why the scalars that occur in supergravity theories belong to a non-linear realisation. However, from the perspective of this paper this it could be viewed as just a consequence of the whole theory being a non-linear realisation. It is known that if one reduces eleven-dimensional supergravity on a torus one finds [14] the group GL$(11-d)$ in d dimensions. From the view point of the non-linear realisation of eleven-dimensional supergravity given in this paper, this is hardly surprising since it is just part of the original GL(11) group of the original theory. However, it would be good to understand the emergence of the exceptional groups from the dimensional reduction of the non-linear realisations given in this paper.

One intriguing feature of the constructions of this paper is that the group is apparently different for each supergravity theory. This would be compatible with the suggestion, in refs. [17, 18], that the full automorphism group is a symmetry of M-theory and that as one goes to the limits of M-theory such as eleven-dimensional supergravity, IIA and IIB theory one finds different contractions of this automorphism group. In fact, it is inevitable that OSp(1/64) is a symmetry of M-theory as this group is the unique extension of the conformal group to include supersymmetry and is in required in both the IIA and eleven-dimensional supergravities. As the IIB supersymmetry algebra can be otained form the IIA supersymmetry algebra by an invertible transformation [28], it is likely that OSp(1/64) is also required in the IIB case. It is interesting to note that by taking this group one automatically encodes all the central charges and the GL(32) automorphism. As such, this group implicitly includes all the branes. Since it includes brane rotating symmetries one would have to restrict the field of the group to ensure it was compatible with the charge quantization conditions.

The maximal supergravities in ten dimensions are the low energy limits of the coresponding string theories. As such, it is perhaps not surprising that they should possess a non-linear realisation in that this has been the traditional role for such formulations. However, it does suggest that there is an alternative formulation of these string theories, perhaps M-theory, in which all the symmetries discussed in this paper, including OSp(1/64) are linearly realised.

From a practical view point, it would be interesting to see if one could use the conformal symmetry, and its superextension, to derive constraints on the Greens functions of the supergravity theories. Such a calculation would utilise our knowledge of

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solving conformal Ward identities with theorems about the behaviour of Greens functions of Goldstone particles. In a sense gravity and supergravity can be thought of as the analogues of the conformally invariant two-dimensional models. One can think of the symmetry of these latter models as being found by starting with the finite-dimensional globally defined conformal group and generating an infinite-dimensional group by taking its closure with the group whose generators are L_2, L_0, L_2. In the theories considered in this paper, one also starts with a finite-dimensional, globally defined, extension of conformal group and generates an infinite-dimensional group by taking its closure with an extension of the affine group.

Acknowledgments

The author would like to thank Bernard Julia for explaining the content of ref. [14], Toine van Proeyen for discussions on superconformal symmetry in eleven dimensions, Arkardy Tseytlin for discussions on the closed bosonic string effective action and George Papadopoulos for commenting on the manuscript. This work was supported in part by the EU network on Integrability, Non-perturbative effects, and Symmetry in Quantum Field theory (FMRX-CT96-0012).

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